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LETTER TO THE EDITOR

Hodge duality and continuum theory of defects

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Abstract. Dual material space–time with defect field is presented in the language of differential forms: one is the strain space–time whose basic equation is the continuity equation for the dislocation 2-form; the other is the stress space–time whose basic equation is the continuity equation for the couple-stress and angular momentum 2-form. Continuity and kinematic equations in each space can be derived by the transformation from p-form to $(p + 1)$-form. Moreover, several constitutive equations can be recognized as the transformation between the p-form of the strain space–time and the $(4 - p)$-form of the stress space–time. These kinematic, continuity and constitutive equations can be interpreted geometrically as Cartan structure equations, Bianchi identities and Hodge duality transformations, respectively.

1. Introduction

Theoretical descriptions of defect field based on differential geometry (e.g. [1]) or gauge theory (e.g. [2]) are referred to as continuum theory of defects. In the last few years, continuum theory of defects has created considerable interest in application to space and planetary sciences such as cosmic strings (e.g. [3]), Einstein–Cartan gravity (e.g. [4]), seismicity (e.g. [5]) and geodesy (e.g. [6]). One important application of continuum theory of defects is to represent the material by two different types of space (e.g. [1, 7]). One is called the strain space whose geometric objects such as metric, torsion and curvature tensors are responsible for strain, dislocation density and disclination density, respectively. The other is called the stress space whose metric, torsion and curvature tensors are responsible for the stress function, couple stress and stress, respectively [7–12]. This symmetric structure implies that the material with defect field has the dual structure of Riemann–Cartan material space. Schaefer [8] was the first to point out this dual point of view and show that continuity equations in strain and stress space can be interpreted geometrically as Bianchi identities. In this case, the kinematic equations in strain space can be interpreted geometrically as the Cartan equation of structure [2].

The strain space and the stress space are not irrelevant to each other. For instance, stress is related to strain through the well known constitutive equation called Hooke's law in the theory of elasticity. The stress function and couple-stress function represent the potential resisting growth of disclination and dislocation densities, respectively (e.g. [7]). These imply that we can link stress and strain space together through several constitutive equations. The problem of how to link them has been highlighted by energy balance aspects such as the variational principle. For instance, Oden and Rebby [13] present dual-complementary variational principles to generalize the Tonti diagram [14], where one intermediate variable (e.g. strain) is paired with the other intermediate variable (e.g. stress) through a constitutive equation (e.g. Hooke's law).

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Table 1. Quantities in strain space–time: F_{3+1}.

	Time-like components	Space-like components
velocity and distortion 1-form: $B^i = v^i\,\mathrm{d}T + \beta^i$	velocity 0-form: v^i	distortion 1-form: $\beta^i = \beta^i_A\,\mathrm{d}x^A$
bend-twist and spin 2-form: $K^i = -\omega^i \wedge \mathrm{d}T + \kappa^i$	spin 1-form: $\omega^i = \omega^i_A\,\mathrm{d}x^A$	bend-twist 2-form: $\kappa^i = \kappa^{iA}\,\mathrm{d}S_A$
dislocation (density and current) 2-form: $D^i = I^i \wedge \mathrm{d}T + \alpha^i$	dislocation current 1-form: $I^i = I^i_A\,\mathrm{d}x^A$	dislocation density 2-form: $\alpha^i = \alpha^{iA}\,\mathrm{d}S_A$
disclination (density and current) 3-form: $\Omega^i = -J^i \wedge \mathrm{d}T + \Theta^i$	disclination current 2-form: $J^i = J^{iA}\,\mathrm{d}S_A$	disclination density 3-form: $\Theta^i = \theta^i\,\mathrm{d}V$

This dual diagram is useful to derive constitutive equations in a systematic way, but lacks the geometric interpretation of the constitutive equations. Meanwhile, constitutive equations in the electromagnetic field, whose geometric structure is similar to that of defect field (e.g. [15]), has already been interpreted geometrically based on the Hodge duality transformations (e.g. [16]).

Thus, we use this duality transformation to achieve clear geometric meanings of the constitutive equations of defect field. Moreover, we express strain and stress space–time in the language of differential form to show the dual structure of Riemann–Cartan material space–time.

2. Strain space–time

In this section, we review the strain space–time based on the work of Edelen and Lagoudas [2]. Let $\{x^1, x^2, x^3, x^4\}$ be the Cartesian coordinates. In this paper, we set $x^4 = cT = \sqrt{E/\rho T}$, where c is the velocity [17], T is the time variable, E is the elastic modulus and ρ is the density of mass. The volume element of three-dimensional space is given by $\mathrm{d}V = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, where the symbol $\wedge$ denotes the wedge product. The oriented surface element of two-dimensional space is given by the inner product: $\mathrm{d}S_A = \langle \partial_A, \mathrm{d}V \rangle$. The $(3 + 1)$-dimensional exterior derivative operator is given by $\mathrm{d} = \mathrm{d}_s + \mathrm{d}T \wedge \partial_t$, where subscript s refers to pure space-differentiation and subscript t to time-differentiation.

Here, let F_{3+1} be the strain space–time. Variables in F_{3+1} expressed by the differential forms are summarized in table 1. For instance, the dislocation (density and current) 2-form D^i is given by

$$D^i = I^i \wedge \mathrm{d}T + \alpha^i \tag{1}$$

where I^i is the dislocation current 1-form and α^i is the dislocation density 2-form. Dislocation density is purely spatial, whereas dislocation current has time dependence. In this paper, we refer to the first term of (1) as time-like components of dislocation 2-form and the second term as space-like components, and we refer to the other quantities in a similar fashion. The basic equation in F_{3+1} is the continuity equation for the dislocation 2-form:

$$\mathrm{d}D^i = \Omega^i \tag{2}$$

where Ω^i is the disclination (density and current) 3-form. This equation can be divided into space-like and time-like components:

$$\mathrm{d}_s\alpha^i = \Theta^i \qquad \text{and} \qquad \partial_t\alpha^i + \mathrm{d}_s I^i = -J^i \tag{3}$$

where Θ^i is the disclination density 3-form and J^i is the disclination current 2-form. Since $\mathrm{d} \cdot \mathrm{d}D^i = 0$, (2) gives the continuity equation for the disclination density 3-form:

$$\mathrm{d}\Omega^i = 0 \tag{4}$$

or

$$\mathrm{d}_s\Theta^i = 0 \qquad \text{and} \qquad \partial_t\Theta^i + \mathrm{d}_s J^i = 0. \tag{5}$$

The general solution of a linear system of inhomogeneous equation (2) is given by the general solution $\mathrm{d}B^i$ of the associated homogeneous system plus a particular solution K^i [2]:

$$D^i = \mathrm{d}B^i + K^i \tag{6}$$

where B^i is the velocity and distortion 1-form and K^i is the bend-twist and spin 2-form. This kinematic equation can be also divided into

$$\alpha^i = \mathrm{d}_s\beta^i + \kappa^i \qquad \text{and} \qquad I^i = \mathrm{d}_s v^i - \partial_t\beta^i - \omega^i \tag{7}$$

where β^i is the distortion 1-form, v^i is the velocity 0-form, κ^i is the bend-twist 2-form and ω^i is the spin 1-form. Finally, from (3) and (6), we have another kinematic equation:

$$\Omega^i = \mathrm{d}K^i \tag{8}$$

or

$$\Theta^i = \mathrm{d}_s\kappa^i \qquad \text{and} \qquad J^i = \mathrm{d}_s\omega^i - \partial_t\kappa^i. \tag{9}$$

In the particular case of $\Omega^i = 0$, (8) and (6) give

$$K^i = \mathrm{d}\phi^i \tag{10}$$

$$D^i = \mathrm{d}\tilde{B}^i \qquad \text{and} \qquad \tilde{B}^i := B^i + \phi^i \tag{11}$$

where ϕ^i is the rotational displacement 1-form. Moreover, if $D^i = 0$, (11) gives

$$\tilde{B}^i = \mathrm{d}u^i \tag{12}$$

where u^i is the displacement 0-form. These results are summarized graphically, in the left-hand side of figure 1. The arrows represent exterior differentiation.

3. Stress space–time

Although the previous strain space is simplified and systematized by a reformulation in a four-dimensional space–time setting [2], the stress space is not four-dimensional. Thus, in this section, we extend the stress space to a (3 + 1)-dimensional setting by introducing stress potential and couple-stress potential [18].

Here, let G_{3+1} be the stress space–time. Variables in G_{3+1} expressed by the differential forms are summarized in table 2. Due to the property of dual transformation (see section 4 for details), time-like components in G_{3+1} are transformed to be space-like in F_{3+1}. Conversely, space-like components in G_{3+1} transform to be time-like in F_{3+1}. Thus, we refer to the time-like components in G_{3+1} as dual space-like components and the space-like components as dual time-like. For instance, stress is dual space-like and momentum is dual time-like. This is more suited to our reality than to regard stress as time-like and momentum as space-like.

The continuity equation for the couple stress and angular momentum 2-form M^i is given by

$$\mathrm{d}M^i = \Sigma^i \tag{13}$$

where Σ^i is the stress and momentum 3-form. As we have seen in section 2, the system of continuity equations and kinematic equations in F_{3+1} are direct consequences of the single statement (2) [2]. In the same way, (13) gives continuity and kinematic equations in G_{3+1} as follows:

$$\mathrm{d}\Sigma^i = 0 \tag{14}$$

$$M^i = \mathrm{d}X^i + C^i \tag{15}$$

$$\Sigma^i = \mathrm{d}C^i \tag{16}$$

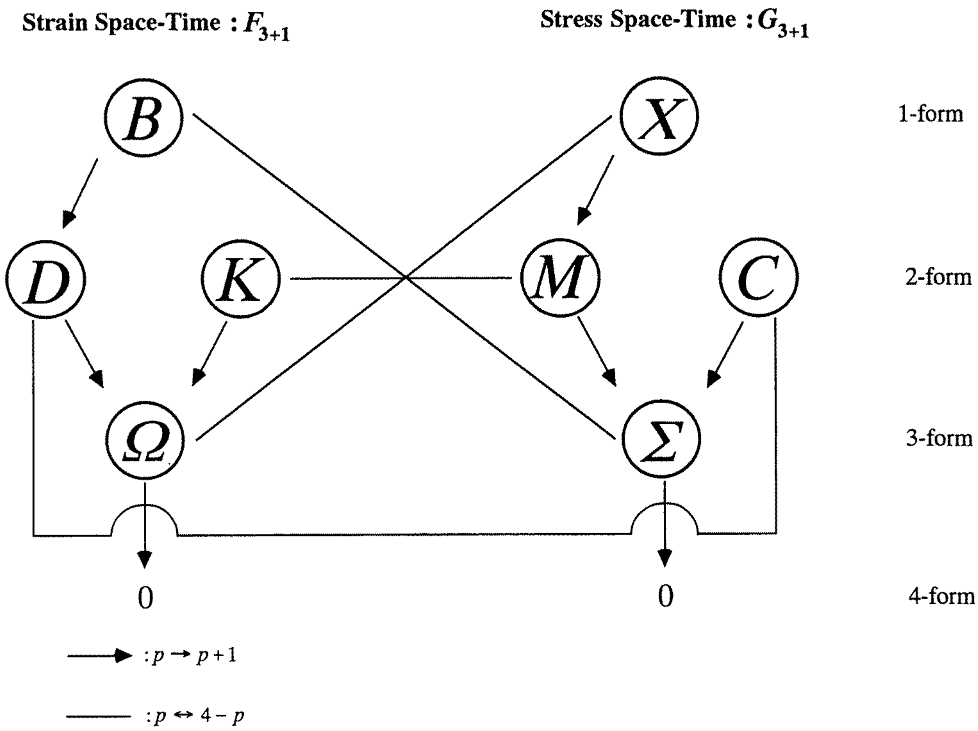

Figure 1. Dual structure of defect field in four-dimensional material space–time.

Table 2. Quantities in stress space–time: G_{3+1}.

	Time-like components (dual space-like components)	Space-like components (dual time-like components)
stress function 1-form: $X^i = \chi^i\,\mathrm{d}T^i + \zeta^i$	stress function 0-form: χ^i	stress potential 1-form: $\zeta^i = \zeta^i_A\,\mathrm{d}x^A$
couple-stress function 2-form: $C^i = -c^i \wedge \mathrm{d}T + \xi^i$	couple-stress function 1-form: $c^i = c^i_A\,\mathrm{d}x^A$	couple-stress potential 2-form: $\xi^i = \xi^{iA}\,\mathrm{d}S_A$
couple-stress and angular momentum 2-form: $M^i = m^i \wedge \mathrm{d}T + a^i$	couple-stress 1-form: $m^i = m^i_A\,\mathrm{d}x^A$	angular momentum 2-form: $a^i = a^{iA}\,\mathrm{d}S_A$
stress and momentum 3-form: $\Sigma^i = -\sigma^i \wedge \mathrm{d}T + P^i$	stress 2-form: $\sigma^i = \sigma^{iA}\,\mathrm{d}S_A$	momentum 3-form: $P^i = p^i\,\mathrm{d}V$

where X^i is the stress and momentum function 1-form and C^i is the couple-stress function 2-form. Dual time-like and dual space-like components of (13)–(16) are given as follows, respectively:

$$\mathrm{d}_s a^i = P^i \quad \text{and} \quad \partial_t a^i + \mathrm{d}_s m^i = -\sigma^i \tag{17}$$

$$\mathrm{d}_s P^i = 0 \quad \text{and} \quad \partial_t P^i + \mathrm{d}_s \sigma^i = 0 \tag{18}$$

$$a^i = \mathrm{d}_s \zeta^i + \xi^i \quad \text{and} \quad m^i = \mathrm{d}_s \chi^i - \partial_t \zeta^i - c^i \tag{19}$$

$$P^i = \mathrm{d}_s \xi^i \quad \text{and} \quad \sigma^i = \mathrm{d}_s c^i - \partial_t \xi^i \tag{20}$$

where notations are summarized in table 2. These results are summarized graphically in the right-hand side of figure 1.

4. Duality and constitutive equations

The vector space of p-vectors and p-forms at any point in an n-dimensional manifold have the dimension: $C_p^n = n!/(n-p)!p!$. Because $C_p^n = C_{n-p}^n$, it is found that at a point there are four spaces which have equal dimension: the vector spaces of p-forms, $(n-p)$-forms, p-vectors and $(n-p)$-vectors [19]. There are two important transformations among these spaces: one is the transformation between the space of p-forms and p-vectors given by the metric tensor; the other is the linear transformation between the space of p-forms and $(n-p)$-vectors given by the Hodge star operator [19]. In this paper, we consider only the linear transformation given by the Hodge star operator on the grounds that the difference between the space of forms and vectors can be neglected in the linear theory of the deformed material-space.

The Hodge star operation is given by

$$* = \frac{(-1)^z}{p!}\varepsilon_{i_{1\ldots}j_{p+1}\cdots j_n} g^{i_{p+1}j_{p+1}} \ldots g^{i_n j_n} \tag{21}$$

where z is the number of negative signs of the metric, $\varepsilon_{i_{1\ldots}j_{p+1}\cdots j_n}$ is the Levi-Civita tensor and $g^{i_{p+1}j_{p+1}} \ldots g^{i_n j_n}$ is the diagonal elements of the metric (e.g. [16] and the references therein). For instance, let us calculate the dual of the velocity and distortion 1-form: $B^i = v^i\,\mathrm{d}T + \beta^i_A\,\mathrm{d}x^A$. From (21) and $x^4 = cT$, we have

$$*B^i = \frac{1}{c}v^i\,\mathrm{d}V - c\beta^{iA}\,\mathrm{d}S_A \wedge \mathrm{d}T \tag{22}$$

where the property of dual transformation should be noted: i.e., the time-like component $v^i\,\mathrm{d}T$ is transformed into the space-like component $v^i\,\mathrm{d}V/c$ and, conversely, $\beta^i_A\,\mathrm{d}x^A$ is transformed into $c\beta^{iA}\,\mathrm{d}S_A \wedge \mathrm{d}T$. Because B^i is a 1-form in F_{3+1}, the associated field of $*B^i$ is a $(4-1=)3$-form in G_{3+1}: that is, the stress and momentum 3-form (see table 2):

$$\Sigma^i = -\sigma^{iA}\,\mathrm{d}S_A \wedge \mathrm{d}T + p^i\,\mathrm{d}V. \tag{23}$$

Then, we express this duality by the following linear relation:

$$\Sigma^i = \sqrt{E\rho} * B^i. \tag{24}$$

From (22) and $c = \sqrt{E/\rho}$, this relation can be expressed explicitly as follows:

$$\begin{aligned}\Sigma^i &= \sqrt{E\rho}\left(\sqrt{\frac{\rho}{E}}v^i\,\mathrm{d}V - \sqrt{\frac{E}{\rho}}\beta^{iA}\,\mathrm{d}S_A \wedge \mathrm{d}T\right)\\ &= \rho v^i\,\mathrm{d}V - E\beta^{iA}\,\mathrm{d}S_A \wedge \mathrm{d}T.\end{aligned} \tag{25}$$

Comparing with (23) and (25), we have the dual time-like and the dual space-like components of (24), respectively:

$$p^i = \rho v^i \qquad \text{and} \qquad \sigma^{iA} = E\beta^{iA}. \tag{26}$$

The first equation is the definition of momentum. The second is the extended Hooke's law in the total strain theory of plasticity [20]. Thus, (24) can be recognized as one of the constitutive equations in four-dimensional material space–time.

In the same way, we have another three dual relations, i.e. four-dimensional constitutive equations (and their dual time-like and space-like components):

$$M^i = e\sqrt{E\rho} * K^i \qquad (a^{iA} = -e\rho\omega^{iA} \text{ and } m^i_A = -eE\kappa^i_A) \tag{27}$$

$$C^i = f\sqrt{E\rho} * D^i \qquad (\xi^{iA} = f\rho I^{iA} \text{ and } c^i_A = fE\alpha^i_A) \tag{28}$$

$$X^i = g\sqrt{E\rho} * \Omega^i \qquad (\zeta^i_A = -g\rho J^i_A \text{ and } \chi^i = -gE\Theta^i) \tag{29}$$

where e, f and g are constants. These results are summarized graphically in figure 1. The solid lines represent the Hodge star operator.

5. Discussions and conclusions

We reformulated the previous three-dimensional stress space [7–12] in a (3 + 1)-dimensional stress space–time (G_{3+1}) to derive the continuity and kinematic equations in G_{3+1}: (13)–(16). Components of the continuity equations (13) and (14) (i.e. (17) and (18)) are in agreement with previous studies (e.g. [7]). Components of the kinematic equations (15) and (16) (i.e. (19) and (20)) are also in agreement with Schaefer [18]. They are summarized by the schematic diagram in figure 1, which shows that continuity and kinematic equations in G_{3+1} can be derived geometrically by the transformation from p-form to $(p + 1)$-form in G_{3+1}.

Next, let us consider these equations of G_{3+1} from the viewpoint of Cartan connections. Edelen and Lagoudas [2] have shown that the kinematic equations of the strain space–time can be mapped into the Cartan equations of structure (see also [7, 21]). By analogy, we express quantities of stress space–time in terms of the geometric objects:

$$X^i = \varphi^i, \qquad C^i = \Gamma^i_j \wedge \varphi^j, \qquad \Sigma^i = \Xi^i_j \wedge \varphi^j - \Gamma^i_j \wedge \Psi^j \qquad \text{and} \qquad M^i = \Psi^i \tag{30}$$

where φ^i is a dual basis 1-form, Γ^i_j is a connection 1-form, Ξ^i_j is a curvature 2-form and Ψ^i is a torsion 2-form. In this case, the kinematic equations (15) and (16) can be rewritten as the Cartan structure equations (e.g. [22, 23]):

$$\Psi^i = \mathrm{d}\varphi^i + \Gamma^i_j \wedge \varphi^j \qquad \text{and} \qquad \Xi^i_j = \mathrm{d}\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j \tag{31}$$

and the continuity equations (13) and (14) can be rewritten as the Bianchi identities (e.g. [22, 23]):

$$\mathrm{d}\Psi^i = \Xi^i_j \wedge \varphi^j - \Gamma^i_j \wedge \Psi^j \qquad \text{and} \qquad \mathrm{d}\Xi^i_j = \Xi^i_k \wedge \Gamma^k_j - \Gamma^i_k \wedge \Xi^k_j. \tag{32}$$

This geometric structure of stress space–time is very similar to that of strain space–time. This dual point of view was first pointed out by Schaefer [8] in the framework of the Riemannian geometry and then extended to the non-Riemannian viewpoint by Stojanovitch [9] and Minagawa [10].

To analyse the geometric meaning of the duality mentioned above, we used the linear Hodge star operator. The result of this analysis is summarized in figure 1, which gives a geometric interpretation of the generalized Tonti diagram [13] and shows that constitutive equations can be derived by the transformation between the p-form of the strain space–time and the $(4 - p)$-form of the stress space–time. Let us compare these derived constitutive equations (24), (27)–(29) with the results of previous studies. Components of (24) (i.e. (26)) have already been derived independently, but it should be noted that these two well known relations can be unified as (24) in four-dimensional space–time. Equation (27) is in agreement with the equation in micropolar theory (e.g. [24] and the references therein). According to micropolar theory, constant e contains a physical property with the dimension of length. The dual space-like components (the second equation) of (28) and (29) have been already pointed out by Amari [7]. On the other hand, the dual time-like components (the first equation) of (28) and (29) are often omitted in the previous paper [7, 10, 12], although Schaefer [18] pointed out the existence of these constitutive equations. In summary, the Hodge duality transformation, which links the strain and stress space, enables us to derive several constitutive equations of previous papers in a systematic way.

In this paper, we suggested the dual material space–time in the language of differential forms: one is the strain space–time, and the other is the stress space–time. By analogy with Edelen and Lagoudas [2], we extended previous three-dimensional stress space to be four-dimensional space–time, whose kinematic and continuity equations are direct consequences of the continuity equation for couple stress and angular momentum 2-form. Moreover, we derived constitutive equations of defect field on the basis of Hodge duality transformation. These results are summarized in figure 1, which shows that the continuity and kinematic equations in each space can be derived by the derivative operator, and the constitutive equations by the Hodge star operator. In this case, kinematic, continuity and constitutive equations can be interpreted geometrically as Cartan structure equations, Bianchi identities and duality transformations, respectively.

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